

A note on random samples of Lie algebras

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Abstract

Recently, Paiva and Teixeira (arXiv:1108.4396) showed that the structure constants of a Lie algebra are the solution of a system of linear equations provided a certain subset of the structure constants are given a-priori. Here it is noted that Lie algebras generated in this way are solvable and their derived subalgebras are Abelian if the system of linear equations considered by Paiva and Teixeira is not degenerate. An efficient numerical algorithm for the calculation of their structure constants is described.

1 Introduction

Recently, Paiva and Teixeira [2011] showed that the structure constants of a finite-dimensional Lie algebra are the solution of a system of linear equations, provided a certain subset of structure constants are given a-priori. We recall that a Lie algebra \mathfrak{L} is a vector space over a field \mathbb{F} equipped with a bilinear product, the Lie bracket $[x, y]$ with $x, y \in \mathfrak{L}$, [see, e.g., Hall 2003; Humphreys 1972; Knapp 2005; Jacobson 1979]. The following discussion is restricted to finite-dimensional Lie algebras over the field of the real ($\mathbb{F} = \mathbb{R}$) or the complex numbers ($\mathbb{F} = \mathbb{C}$). The Lie bracket satisfies $[x, x] = 0$ for all $x \in \mathfrak{L}$ and the Jacobi identity

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0 \quad (1)$$

for all $x, y, z \in \mathfrak{L}$. The N -dimensional Lie algebra \mathfrak{L} is completely characterized by the coordinates (known as structure constants) $\mathbf{f}\{i, j, k\}$ of the Lie bracket product with respect to the basis g_i [see, e.g., Hall 2003; Humphreys 1972; Jacobson 1979],

$$[g_i, g_j] = \sum_{k=1}^N \mathbf{f}\{i, j, k\} g_k \quad i, j = 1, \dots, N \quad . \quad (2)$$

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Here and in the following, the notation $\mathbf{X}\{i_1, \dots, i_D\}$ is used to identify a specific element of the D -dimensional matrix \mathbf{X} indexed by D positive integers i_1, \dots, i_D . Vectors are represented as 2-dimensional matrices with one singleton dimension. Specifically, $i_1 = 1$ and $i_2 = 1, \dots, N$ describes a row vector $\vec{\mathbf{X}} \equiv (\mathbf{X}\{1, 1\}, \dots, \mathbf{X}\{1, N\})$ and $i_1 = 1, \dots, N$ and $i_2 = 1$ its transpose, a column vector $\vec{\mathbf{X}}^T \equiv (\mathbf{X}\{1, 1\}, \dots, \mathbf{X}\{N, 1\})$.

Paiva and Teixeira [2011] observed that the Jacobi identity (1), expressed in terms of the structure constants $\mathbf{f}\{i, j, k\}$, is

$$J_{i,j,k,m}^A + J_{i,j,k,m}^B + J_{i,j,k,m}^C = 0 \quad (3)$$

with

$$\begin{aligned} J_{i,j,k,m}^A &\equiv \sum_{l=1}^N \mathbf{f}\{i, j, l\} \mathbf{f}\{k, l, m\} \\ J_{i,j,k,m}^B &\equiv \sum_{l=1}^N \mathbf{f}\{k, i, l\} \mathbf{f}\{j, l, m\} \\ J_{i,j,k,m}^C &\equiv \sum_{l=1}^N \mathbf{f}\{j, k, l\} \mathbf{f}\{i, l, m\} \end{aligned} \quad (4)$$

and $i, j, k, m = 1, \dots, N$, represents a system of linear equations in unknowns

$$\mathbf{f}\{2 \leq i \leq N, i+1 \leq j \leq N, 1 \leq k \leq N\} \quad (5)$$

if the index i in (3) is fixed at $i = 1$. Since $\mathbf{f}\{i, j, k\} = -\mathbf{f}\{j, i, k\}$, the three summands in (3) are related,

$$\begin{aligned} J_{i,j,k,m}^A &= -J_{j,i,k,m}^A = -J_{k,j,i,m}^C = -J_{i,k,j,m}^B \\ J_{i,j,k,m}^B &= -J_{j,i,k,m}^C = -J_{k,j,i,m}^B = -J_{i,k,j,m}^A \\ J_{i,j,k,m}^C &= -J_{j,i,k,m}^B = -J_{k,j,i,m}^A = -J_{i,k,j,m}^C \end{aligned} \quad (6)$$

and the range of i, j and k in (3) can be restricted to $1 \leq i < j < k \leq N$ without omitting linear independent equations. Paiva and Teixeira [2011] considered the system of equations

$$\begin{aligned} J_{i=1,j,k,m}^A + J_{i=1,j,k,m}^B + J_{i=1,j,k,m}^C &= \sum_{l=1}^N \mathbf{f}\{i=1, j, l\} \mathbf{f}\{k, l, m\} \\ &\quad - \mathbf{f}\{i=1, k, l\} \mathbf{f}\{j, l, m\} + \mathbf{f}\{i=1, l, m\} \mathbf{f}\{j, k, l\} \\ &= 0 \end{aligned} \quad (7)$$

with

$$j = 2, \dots, N \quad k = j+1, \dots, N \quad m = 1, \dots, N \quad (8)$$

and $\mathbf{f}\{i=1, j, k\}$ taken as known parameters.

The number of individual linear equations in (7),

$$N \sum_{j=2}^N \left(\sum_{k=j+1}^N 1 \right) = N \sum_{j=2}^N (N-j) = N(N-1)(N-2)/2 \quad (9)$$

matches the number of unknowns (5). Thus, a unique solution exists provided the a-priori parameters $\mathbf{f}\{i=1, j, k\}$ are chosen such, that the system of linear equations is non-degenerate [see, e.g, Strang 2009]. We assume that this condition is fulfilled.

In the following section an efficient algorithm for calculating $\mathbf{f}\{i, j, k\}$ is described bypassing the task of solving the system of linear equations in (7). The appendix includes a computer implementation of the algorithm. Second, it is shown that Lie algebras randomly generated in this way belong to the class of solvable Lie algebras and their derived subalgebras $[\mathfrak{L}, \mathfrak{L}]$ are Abelian. The algorithm described by Paiva and Teixeira [2011] may therefore be used to generate random solvable Lie algebras [as regards random samples of nilpotent Lie algebras, see also Luks 1977].

2 Generating a random Lie algebra

As will be evident in the following, it is convenient to rewrite the problem of solving (7), in terms of the $N \times N$ matrices \mathbf{A}_k with matrix elements

$$\mathbf{A}_k\{i, j\} = \mathbf{f}\{k, j, i\} \quad . \quad (10)$$

\mathbf{A}_k ($k = 1, \dots, N$) are the adjoint representation of \mathfrak{L} , [see, e.g., Hall 2003; Humphreys 1972] and (2) translates into the matrix equation

$$\begin{aligned} [\mathbf{A}_i, \mathbf{A}_j] &= \mathbf{A}_i \cdot \mathbf{A}_j - \mathbf{A}_j \cdot \mathbf{A}_i = \sum_{k=1}^N \mathbf{f}\{i, j, k\} \mathbf{A}_k \\ &= \sum_{k=1}^N \mathbf{A}_i\{k, j\} \mathbf{A}_k \quad . \end{aligned} \quad (11)$$

In terms of the adjoint representation the a-priori parameters are precisely the elements of the matrix, $\mathbf{A}_{k=1}$. According to Paiva and Teixeira [2011] all matrices of the adjoint representation can be calculated from $\mathbf{A}_{k=1}$ provided the constraints discussed above are satisfied.

We introduce an a-priori matrix \mathbf{P} containing $N(N-1)$ parameters

$$c \mathbf{A}_1 \equiv \mathbf{P} \equiv \begin{pmatrix} 0 & \mathbf{P}\{1, 2\} & \dots & \mathbf{P}\{1, N\} \\ \vdots & \vdots & & \vdots \\ 0 & \mathbf{P}\{N, 2\} & \dots & \mathbf{P}\{N, N\} \end{pmatrix} \quad (12)$$

with some scalar factor $c \in \mathbb{F}$. Since $\mathbf{f}\{i, i, k\} = 0$, the first column of $\mathbf{A}_{k=1}$ is zero and, therefore, the rank of \mathbf{P} , $\text{Rk}(\mathbf{P})$, is at most $N-1$; in the following it is assumed that the

parameters $\mathbf{P}\{i, j\}$ are chosen such that

$$\text{Rk}(\mathbf{P}) = N - 1 \quad . \quad (13)$$

The null space of \mathbf{P} is therefore one-dimensional and $\vec{\mathbf{n}}$, the normalized null vector of \mathbf{P} , is unique (up to its sign) with

$$\vec{\mathbf{n}} \cdot \mathbf{P} = (0, \dots, 0) \quad . \quad (14)$$

Following Paiva and Teixeira [2011] we expect the elements of the matrices $\mathbf{A}_{k \neq 1}$ to solve a system of linear equations and introduce the *Ansatz*

$$\mathbf{A}_k = \mathbf{P} \cdot \mathbf{T}_k \quad k = 1, \dots, N \quad (15)$$

with \mathbf{T}_k given by

$$\mathbf{T}_k = \mathbf{n}\{k\} \mathbb{1} - \vec{\mathbf{e}}_k^T \cdot \vec{\mathbf{n}} \quad . \quad (16)$$

Here, $\mathbb{1}$ is the $N \times N$ unit matrix and $\vec{\mathbf{e}}_k$ is the row vector with N elements,

$$\vec{\mathbf{e}}_k = (0, \dots, 0, \underset{k}{1}, 0, \dots, 0) \quad (17)$$

I.e. $\mathbf{e}_k\{k\} = 1$ and $\mathbf{e}_k\{i\} = 0$ for all $i \neq k$. Note that $\vec{\mathbf{n}} \cdot \vec{\mathbf{e}}_k^T$ is a scalar, whereas $\vec{\mathbf{e}}_k^T \cdot \vec{\mathbf{n}}$ evaluates to an $N \times N$ matrix.

It is convenient to choose

$$c \equiv \frac{1}{\mathbf{n}\{1\}} \quad (18)$$

in (12) assuming $\mathbf{n}\{1\} \neq 0$. The choice (18) allows us to write both, $\mathbf{A}_{k=1}$ and $\mathbf{A}_{k \neq 1}$ in the form of (15), since

$$\begin{aligned} \mathbf{A}_{k=1} &= \mathbf{P} \cdot \mathbf{T}_{k=1} \\ &= \mathbf{P} \cdot \begin{pmatrix} 0 & -\mathbf{n}\{2\} & \dots & \dots & -\mathbf{n}\{N\} \\ 0 & \mathbf{n}\{1\} & 0 & \dots & 0 \\ \vdots & 0 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & \dots & 0 & \mathbf{n}\{1\} \end{pmatrix} \\ &= \mathbf{n}\{1\} \mathbf{P} \end{aligned} \quad (19)$$

and taking into account the vanishing first column of \mathbf{P} . In order to prove that *Ansatz* (16) indeed constitutes an adjoint representation of \mathfrak{L} , the matrices $\mathbf{A}_k = \mathbf{P} \cdot \mathbf{T}_k$ must be shown to satisfy (11). We proceed by first calculating the left-hand side of (11), the Lie bracket of \mathbf{A}_i and \mathbf{A}_j ,

$$\begin{aligned} [\mathbf{A}_i, \mathbf{A}_j] &= \mathbf{A}_i \cdot \mathbf{A}_j - \mathbf{A}_j \cdot \mathbf{A}_i \\ &= \mathbf{P} \cdot (\mathbf{n}\{i\} \mathbb{1} - \vec{\mathbf{e}}_i^T \cdot \vec{\mathbf{n}}) \cdot \mathbf{P} \cdot (\mathbf{n}\{j\} \mathbb{1} - \vec{\mathbf{e}}_j^T \cdot \vec{\mathbf{n}}) \\ &\quad - \mathbf{P} \cdot (\mathbf{n}\{j\} \mathbb{1} - \vec{\mathbf{e}}_j^T \cdot \vec{\mathbf{n}}) \cdot \mathbf{P} \cdot (\mathbf{n}\{i\} \mathbb{1} - \vec{\mathbf{e}}_i^T \cdot \vec{\mathbf{n}}) \\ &= \mathbf{n}\{i\} \mathbf{P}^2 \cdot (\mathbf{n}\{j\} \mathbb{1} - \vec{\mathbf{e}}_j^T \cdot \vec{\mathbf{n}}) \\ &\quad - \mathbf{n}\{j\} \mathbf{P}^2 \cdot (\mathbf{n}\{i\} \mathbb{1} - \vec{\mathbf{e}}_i^T \cdot \vec{\mathbf{n}}) \end{aligned} \quad (20)$$

on account of (14) and, finally,

$$\begin{aligned}
[\mathbf{A}_i, \mathbf{A}_j] &= \mathbf{n}\{i\} \mathbf{n}\{j\} \mathbf{P}^2 - \mathbf{n}\{i\} \mathbf{P}^2 \cdot \vec{\mathbf{e}}_j^T \cdot \vec{\mathbf{n}} \\
&\quad - \mathbf{n}\{j\} \mathbf{n}\{i\} \mathbf{P}^2 + \mathbf{n}\{j\} \mathbf{P}^2 \cdot \vec{\mathbf{e}}_i^T \cdot \vec{\mathbf{n}} \\
&= \mathbf{P}^2 \cdot (\mathbf{n}\{j\} \vec{\mathbf{e}}_i^T - \mathbf{n}\{i\} \vec{\mathbf{e}}_j^T) \cdot \vec{\mathbf{n}}.
\end{aligned} \tag{21}$$

Second, the right-hand side of (11) is found to be

$$\begin{aligned}
\sum_{k=1}^N \mathbf{A}_i\{k, j\} \mathbf{A}_k &= \sum_{k=1}^N (\mathbf{P} (\mathbf{n}\{i\} \mathbb{1} - \vec{\mathbf{e}}_i^T \cdot \vec{\mathbf{n}})) \{k, j\} \mathbf{P} \cdot (\mathbf{n}\{k\} \mathbb{1} - \vec{\mathbf{e}}_k^T \cdot \vec{\mathbf{n}}) \tag{22} \\
&= \sum_{k=1}^N (\mathbf{P} (\mathbf{n}\{i\} \mathbb{1} - \vec{\mathbf{e}}_i^T \cdot \vec{\mathbf{n}}) \cdot \vec{\mathbf{e}}_j^T) \{k\} \mathbf{P} \cdot (\mathbf{n}\{k\} \mathbb{1} - \vec{\mathbf{e}}_k^T \cdot \vec{\mathbf{n}}) \\
&= \mathbf{P} \cdot \sum_{k=1}^N (\mathbf{P} (\mathbf{n}\{i\} \mathbb{1} - \vec{\mathbf{e}}_i^T \cdot \vec{\mathbf{n}}) \cdot \vec{\mathbf{e}}_j^T) \{k\} \mathbf{n}\{k\} \\
&\quad - \mathbf{P} \cdot \sum_{k=1}^N (\mathbf{P} (\mathbf{n}\{i\} \mathbb{1} - \vec{\mathbf{e}}_i^T \cdot \vec{\mathbf{n}}) \cdot \vec{\mathbf{e}}_j^T) \{k\} \vec{\mathbf{e}}_k^T \cdot \vec{\mathbf{n}} \\
&= \mathbf{P} \cdot \vec{\mathbf{n}} \cdot \mathbf{P} (\mathbf{n}\{i\} \mathbb{1} - \vec{\mathbf{e}}_i^T \cdot \vec{\mathbf{n}}) \cdot \vec{\mathbf{e}}_j^T \\
&\quad - \mathbf{P} \cdot \left(\sum_{k=1}^N (\mathbf{P} \cdot (\mathbf{n}\{i\} \mathbb{1} - \vec{\mathbf{e}}_i^T \cdot \vec{\mathbf{n}}) \cdot \vec{\mathbf{e}}_j^T) \{k\} \vec{\mathbf{e}}_k^T \right) \cdot \vec{\mathbf{n}}.
\end{aligned}$$

The first term vanishes owing to (14) and the second simplifies with (17) to

$$\begin{aligned}
\sum_{k=1}^N \mathbf{A}_i\{k, j\} \mathbf{A}_k &= -\mathbf{P} \cdot (\mathbf{P} \cdot (\mathbf{n}\{i\} \mathbb{1} - \vec{\mathbf{e}}_i^T \cdot \vec{\mathbf{n}}) \cdot \vec{\mathbf{e}}_j^T) \cdot \vec{\mathbf{n}} \tag{23} \\
&= \mathbf{P}^2 \cdot (\mathbf{n}\{j\} \vec{\mathbf{e}}_i^T - \mathbf{n}\{i\} \vec{\mathbf{e}}_j^T) \cdot \vec{\mathbf{n}}
\end{aligned}$$

which equals the right-hand side of (21). We conclude that the matrices

$$\mathbf{A}_k = \mathbf{P} \cdot (\mathbf{n}\{k\} \mathbb{1} - \vec{\mathbf{e}}_k^T \cdot \vec{\mathbf{n}}) \quad k = 1, \dots, N \tag{24}$$

form an adjoint representation of \mathfrak{L} . Furthermore, the proof of *Ansatz* (15) implies that the adjoint representation matrices of \mathfrak{L} (and with the help of (10) also all its structure constants) can be uniquely determined, once the matrix \mathbf{P} with its $N(N-1)$ a-priori parameters is given, provided $\text{Rk}(\mathbf{P}) = N-1$ and $\mathbf{n}\{1\} \neq 0$.

3 Derived series and lower central series

Knowing the matrices of the adjoint representation (24) it is straightforward to calculate the derived series of \mathfrak{L} [Humphreys 1972]. With the abbreviation

$$\vec{\mathbf{m}}_{j,i}^T \equiv \mathbf{n}\{j\} \cdot \vec{\mathbf{e}}_i^T - \mathbf{n}\{i\} \cdot \vec{\mathbf{e}}_j^T \tag{25}$$

from (21) it follows

$$\begin{aligned}
[[\mathbf{A}_i, \mathbf{A}_j], [\mathbf{A}_k, \mathbf{A}_l]] &= [\mathbf{P}^2 \cdot \vec{\mathbf{m}}_{j,i}^T \cdot \vec{\mathbf{n}}, \mathbf{P}^2 \cdot \vec{\mathbf{m}}_{l,k}^T \cdot \vec{\mathbf{n}}] \\
&= \mathbf{P}^2 \cdot \vec{\mathbf{m}}_{j,i}^T \cdot \vec{\mathbf{n}} \cdot \mathbf{P}^2 \cdot \vec{\mathbf{m}}_{l,k}^T \cdot \vec{\mathbf{n}} \\
&\quad - \mathbf{P}^2 \cdot \vec{\mathbf{m}}_{l,k}^T \cdot \vec{\mathbf{n}} \cdot \mathbf{P}^2 \cdot \vec{\mathbf{m}}_{j,i}^T \cdot \vec{\mathbf{n}} \\
&= 0
\end{aligned} \tag{26}$$

for all $i, j, k, l = 1, \dots, N$ owing to (14). Thus, the Lie algebra \mathfrak{L} is solvable and the derived subalgebra $[\mathfrak{L}, \mathfrak{L}]$ is Abelian.

Alternatively, solvability of \mathfrak{L} may be proved using Cartan's criterion, i.e. the fact that the Killing form $K(x, y)$ with $x \in \mathfrak{L}$ and $y \in [\mathfrak{L}, \mathfrak{L}]$ is identically zero if and only if \mathfrak{L} is solvable [Knapp 2005]. We find

$$\begin{aligned}
\text{Tr}(\mathbf{A}_i \cdot [\mathbf{A}_j, \mathbf{A}_k]) &= \text{Tr}([\mathbf{A}_j, \mathbf{A}_k] \cdot \mathbf{A}_i) \\
&= \text{Tr}(\mathbf{P}^2 \cdot \vec{\mathbf{m}}_{k,j}^T \cdot \vec{\mathbf{n}} \cdot \mathbf{P} \cdot (\mathbf{n}\{i\} \mathbb{1} - \vec{\mathbf{e}}_i^T \cdot \vec{\mathbf{n}})) \\
&= \text{Tr}(0) = 0
\end{aligned} \tag{27}$$

for all $i, j, k = 1, \dots, N$ owing to (14).

The lower central series [Humphreys 1972], on the other hand, is

$$[\mathbf{A}_{i_L}, \dots, [\mathbf{A}_{i_1}, [\mathbf{A}_j, \mathbf{A}_k]] \dots] = \mathbf{n}\{i_L\} \dots \mathbf{n}\{i_1\} \mathbf{P}^{L+2} \cdot \vec{\mathbf{m}}_{k,j}^T \cdot \vec{\mathbf{n}} \tag{28}$$

which follows from

$$\begin{aligned}
[\mathbf{A}_i, [\mathbf{A}_j, \mathbf{A}_k]] &= [\mathbf{P} \cdot (\mathbf{n}\{i\} \mathbb{1} - \vec{\mathbf{e}}_i^T \cdot \vec{\mathbf{n}}), \mathbf{P}^2 \cdot \vec{\mathbf{m}}_{k,j}^T \cdot \vec{\mathbf{n}}] \\
&= \mathbf{n}\{i\} \mathbf{P}^3 \cdot \vec{\mathbf{m}}_{k,j}^T \cdot \vec{\mathbf{n}}.
\end{aligned} \tag{29}$$

If the parameter matrix \mathbf{P} is not nilpotent, the right-hand side of (28) does not vanish. Accordingly, the lower central series of \mathfrak{L} will not terminate and \mathfrak{L} not be nilpotent.

If, however, \mathbf{P} is chosen to be strictly upper-triangular (upper-triangular with zeros on the main diagonal), then $\mathbf{P}^N = 0$. In this case the series (28) terminates and the corresponding Lie algebra is nilpotent, too.

4 Concluding remarks

We have shown that the adjoint representation of a random, solvable Lie algebra \mathfrak{L} can be obtained from a (real or complex) parameter matrix \mathbf{P}

$$\mathbf{A}_k = \mathbf{P} \cdot (\mathbf{n}\{k\} \mathbb{1} - \vec{\mathbf{e}}_k^T \cdot \vec{\mathbf{n}}) \quad k = 1, \dots, N \tag{30}$$

where $\vec{\mathbf{n}}$ denotes the null vector of \mathbf{P} . This result relies on the assumption, that $\text{Rk}(\mathbf{P}) = N - 1$ and $\mathbf{n}\{1\} \neq 0$. We find that \mathfrak{L} is a solvable Lie algebra with an Abelian derived subalgebra; if \mathbf{P} is strictly upper-triangular, then the Lie algebra \mathfrak{L} is nilpotent as well.

Relation (30) implies that all matrices \mathbf{A}_k share the same null space. Furthermore, it is worthwhile to note that

$$\mathbf{T}_j \cdot \mathbf{T}_k = \mathbf{n}\{j\} \mathbf{T}_k \quad ; \quad (31)$$

left-multiplication by \mathbf{P} yields

$$\mathbf{A}_j \cdot \mathbf{T}_k = \mathbf{n}\{j\} \mathbf{A}_k \quad . \quad (32)$$

I.e. the matrix \mathbf{T}_k not only generates the adjoint representation matrix \mathbf{A}_k from the a-priori matrix \mathbf{P} , it also may be used to transform any adjoint representation matrix into \mathbf{A}_k .

References

- Eaton, John W., David Bateman, and Søren Hauberg [2008]. *GNU Octave manual version 3*. Network Theory Ltd. ISBN: 095461206X.
- Hall, Brian C. [2003]. *Lie groups, Lie algebras and representations*. Springer-Verlag, New York, USA. ISBN: 0-387-40122-9.
- Humphreys, James E. [1972]. *Introduction to Lie algebras and representation theory*. Springer-Verlag, New York, USA. ISBN: 0-387-90053-5.
- Jacobson, Nathan [1979]. *Lie algebras*. Dover Publications, New York NY, USA. ISBN: 0-486-63832-4.
- Knapp, Anthony W. [2005]. *Lie groups beyond an introduction*. Birkhäuser, Boston MA, USA. ISBN: 0-8176-4259-5.
- Luks, Eugene M. [1977]. “What is the typical nilpotent Lie algebra?” In: *Computers in the Study of Nonassociative Rings and Algebras*. Ed. by Robert E. Beck and Bernard Kolman. Academic Press, New York, USA, pp. 189–208. ISBN: 0-12-083850-8.
- Paiva, F. M. and A. F. F. Teixeira [2011]. *Algorithm for structure constants*. arXiv:1108.4396 [physics]. URL: www.arxiv.org/abs/1108.4396.
- Strang, Gilbert [2009]. *Introduction to linear algebra*. Wellesley-Cambridge Press, Wellesley MA, USA. ISBN: 978-09802327-14.

A Computer implementation

An implementation of the algorithm described in section 2 using the Octave programming language [Eaton, Bateman, and Hauberg 2008] is reproduced below.

```
function [adjRep, structConst] = randsolvableliealg( nofDim, allowCplx)
```

```

if nargin < 1 || isempty( nofDim), nofDim = 3; end
if nargin < 2 || isempty( allowCplx),
    allowCplx = false;
end
aPriori = randn( nofDim, nofDim-1);
if allowCplx,
    aPriori = aPriori + 1i * randn( nofDim, nofDim-1);
end
% note:
% aPriori = aPriori - tril( aPriori);
% generates a random nilpotent Lie algebra
if rank( aPriori) ~= nofDim-1,
    error( [upper( mfilename) ...
        ': null space of parameter matrix not one-dimensional.'])
end
pMat = zeros( nofDim, nofDim);
% first column of 'pMat' is zero and ...
pMat(:,2:end) = aPriori;
% ... thus null space of matrix 'aPriori' is one-dimensional
nVct = transpose( null( transpose( aPriori)));
assert( size( nVct, 1) == 1)
unitMat = eye( nofDim);
adjRep = zeros( nofDim, nofDim, nofDim) * NaN;
for k = 1:nofDim,
    adjRep(:,:,k) = pMat * (nVct(k) * unitMat - unitMat(:,k) * nVct);
end
if nargin > 1,
    structConst = permute( adjRep, [3, 2, 1]);
end
end
end

```

On a standard personal computer (2 GHz Intel(R) Core(TM)2 Duo processor, 4 GByte memory) running a Linux operating system with GNU Octave (version 3.8.1) it takes about 0.3 seconds to generate a solvable Lie algebra with dimension $N = 100$. The solution for $N = 500$ requires about 40 seconds.